Transport in a Magnetic Field. III. Application to Spin Waves in Alkali Metals*

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The transport equation derived in the preceding paper is compared to Silin's and applied to the problem of interpreting the spin-wave spectrum of weakly magnetized sodium. Several schemes for parametrizing the conduction-electron interactions are considered; it is shown that the experiments do not yet favor any particular scheme, and that therefore the nature of the interactions is still undetermined. An uncertainty-principle argument is offered to suggest that both transport equations are of suspect validity in the range of frequencies and wavelengths so far experimentally explored.

I. INTRODUCTION

SPIN waves in weakly magnetized alkali metals have been observed as satellite resonances to conduction-electron spin resonance.[1] This confirms an early, general prediction of the existence of such spin waves as given by Herring and Kittel.[2] The details of the interactions between conduction electrons are of crucial importance in the determination of the qualitative as well as quantitative aspects of the spin-wave frequency-wavelength relation. In the complete absence of interactions, for example, there are no spin waves at all; if the interactions be attractive, the dispersion in the frequency range so far explored would be linear and nearly independent of the weak magnetization[3] whereas repulsive interactions, as in alkalis, give a quadratic spectrum, $\omega \propto q^2+q_0^2$, which merges with the single-particle continuum at progressively smaller q as the magnetization goes to zero. In conjunction with the report of the experimental discovery, Platzman and Wolff[4] have presented the results of an analysis of the spin-wave spectrum in which particular attention is paid to the experimental circumstances.

The spectrum depends strongly on the angle between external field and propagation direction, and the measurement of this anisotropy can provide information on the interactions. The theory of Platzman and Wolff takes as its starting point a transport equation due to Silin,[5] the solutions to which determine the frequency and wavelength dependence of the spin susceptibility as functions of the interaction parameters. The structure of the susceptibility function in turn gives the spin-wave spectrum.

The purpose of our discussion is a reexamination of the interpretation of the experimental results in terms of parameters relating to the electron interactions.

Our remarks have three major aspects. First, we doubt the transport equation used in Ref. 4. Unfortunately, Silin's derivation[5,6] of this equation is not presented in sufficient detail for specific criticisms to be made. We have presented a detailed derivation of our proposed alternative transport equation in two papers preceding this.[7,8] We shall cite the results of that derivation here and contrast it term by term with Silin's.

Second, the use of any transport equation to obtain information about particle interactions requires auxiliary assumptions about the class of functional forms which may be used to describe such interactions. That is, the interaction function must be parametrized. We shall show that this may be done in a variety of more or less equally plausible ways, all quite consistent with the experiments.[1]

Finally, we argue that neither our equation[8] nor Silin's[5] may apply in this range of wavelengths and frequencies, that the attempt to apply them anyway may represent a violation of the uncertainty relation between noncommuting observables. It will be clear, however, that the range of clear validity of the transport equation is experimentally accessible.

We proceed, then, to our interpretation of the experiments based on semiclassical Boltzmann-like equations because (1) the equations are not manifestly invalid and the limitations we cite may be only of the derivation and not its result; (2) many authors have used semiclassical equations, and so they represent a conventional approximation whose predictions are worthy of recording; (3) the numbers produced by this approximation outside its range of clear validity may eventually be compared to those obtained in the accessible, valid regime.

In Sec. II, we effect the detailed comparison of our equation and Silin's. In Sec. III, we consider various possible parametrizations of the interaction potential of the electrons and the values of the parameters the experiments demand. An interesting result of this section is that, using our transport equation, only one parameter of the interaction can be determined from the experiment of Ref. 1, rather than the two parameters found in Ref. 4, and that very parameter occurs in the

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theory of the Pauli paramagnetic susceptibility. In Sec. IV, we discuss the failure of the semiclassical approximation to account properly for the uncertainty principle.

II. TRANSPORT EQUATIONS

A Boltzmann form of equation relating electron-spin magnetization density to its time and space derivatives has been given by Silin.[5] Platzman and Wolff[4] use this formula to obtain the satellite spin-wave spectrum,

$$\frac{\partial g}{\partial t}+\left[\mathbf{v}\cdot\boldsymbol{\nabla}+\frac{e}{\hbar c}\mathbf{v}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{k}}+i\Omega_0\right](g+\delta\epsilon_2)$$
$$=\tfrac{1}{2}\gamma_0(\mathbf{v}\cdot\nabla+i\Omega_0)h^+. \quad (1)$$

In this, $\sum_{\mathbf{k}}-(\partial f_0/\partial E_{\mathbf{k}})g$ is the total rf spin magnetization density, $\mathbf{v}$ is the electron velocity, $\mathbf{B}$ the applied magnetic field, Ω_0 a "renormalized" spin-precession frequency,

$$\delta\epsilon_2=\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\delta[E_{\mathbf{k}}-E_F]g(\mathbf{k}'), \quad (2)$$

$V(\mathbf{k},\mathbf{k}')$ an effective electron-interaction potential function and h^+ a circularly polarized rf driving field. We have allowed the electron lifetimes to be infinite since we shall not be interested in any effects arising from the failure of this approximation. These effects broaden the observed resonances but do not shift them substantially.[1]

In Ref. 8 we obtained an equation for the $\mathbf{q}$th Fourier component of the magnetization density.

$$i\hbar\frac{\partial\mathfrak{M}_{\mathbf{k}}}{\partial t}+\left\{\epsilon_{\mathbf{k}\uparrow}-\epsilon_{\mathbf{k}+\mathbf{q}\downarrow}-i\hbar\frac{e}{mc}\mathbf{B}\times(\mathbf{k}+\mathbf{q}/2)\cdot\boldsymbol{\nabla}_{\mathbf{k}}\right\}\mathfrak{M}_{\mathbf{k}}$$
$$=\tfrac{1}{2}\mathfrak{N}_{\mathbf{k}}\gamma_0h^+-\sum_{\mathbf{k}'}V(\mathbf{k}-\mathbf{k}')(\mathfrak{N}_{\mathbf{k}}\mathfrak{M}_{\mathbf{k}'}-\mathfrak{N}_{\mathbf{k}'}\mathfrak{M}_{\mathbf{k}}). \quad (3)$$

In this,

$$\mathfrak{N}_{\mathbf{k}}\equiv n_{\mathbf{k}+\mathbf{q}\downarrow}-n_{\mathbf{k}\uparrow}, \quad (4)$$
$$\mathfrak{M}_{\mathbf{k}}=b(\mathbf{k})\mathfrak{N}_{\mathbf{k}}. \quad (5)$$

The function $b(\mathbf{k})$ measures the response of each electron to the spin-wave disturbance. We have used $\frac{1}{2}\gamma_0h^+$ in (3); the corresponding parameter in Ref. 8 was c_0.

The last terms in (3) are the exchange energies. If we define $\mathcal{E}_{\mathbf{k},\sigma}$ as the "one-particle energy," we have

$$\mathcal{E}_{\mathbf{k},\sigma}=\epsilon_{\mathbf{k},\sigma}-\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')n_{\mathbf{k}',\sigma}, \quad (6)$$

and obtain

$$i\hbar\frac{\partial\mathfrak{M}_{\mathbf{k}}}{\partial t}+\{\mathcal{E}_{\mathbf{k}\uparrow}-\mathcal{E}_{\mathbf{k}+\mathbf{q}\downarrow}-i\hbar\dot{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\}\mathfrak{M}_{\mathbf{k}}$$
$$=\tfrac{1}{2}\mathfrak{N}_{\mathbf{k}}\gamma_0h+\mathfrak{N}_{\mathbf{k}}\sum_{\mathbf{k}'}V(\mathbf{k}\mathbf{k}')\mathfrak{M}_{\mathbf{k}'}, \quad (7)$$

$$\dot{\mathbf{k}}\equiv\frac{e}{mc}\mathbf{B}\times\mathbf{k}.$$

This is our master equation and is almost identical to Eq. (1). This assertion requires demonstration, however. Let $\mathfrak{M}_{\mathbf{k}}=(\partial f_0/\partial t)g_{\mathbf{k}}$. The energy difference $\mathcal{E}_{\mathbf{k}+\mathbf{q}\downarrow}-\mathcal{E}_{\mathbf{k}\uparrow}$ may be written $\mathcal{E}_{\mathbf{k}+\mathbf{q}\downarrow}-\mathcal{E}_{\mathbf{k}\downarrow}+\mathcal{E}_{\mathbf{k}\downarrow}-\mathcal{E}_{\mathbf{k}\uparrow}$. The last two terms of this are

$$\gamma_0B_0+\sum_{\mathbf{k}'}V(\mathbf{k}\mathbf{k}')(n_{\mathbf{k}'\uparrow}-n_{\mathbf{k}'\downarrow})=\gamma_0B_{\text{eff}}, \quad (8)$$

and the first two are clearly $\mathbf{q}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}=\mathbf{q}\cdot\hbar\mathbf{v}_{\mathbf{k}}$ if $q\ll k_F$. Now $\boldsymbol{\nabla}_{\mathbf{k}}$ operating on $\mathfrak{M}_{\mathbf{k}}$ gives

$$\boldsymbol{\nabla}_{\mathbf{k}}\mathfrak{M}_{\mathbf{k}}=\frac{\partial f_0}{\partial\mathcal{E}}\nabla_{\mathbf{k}}g+\left(\frac{\partial^2f_0}{\partial\mathcal{E}^2}\nabla_{\mathbf{k}}\mathcal{E}\right)g. \quad (9)$$

But since $\dot{\mathbf{k}}=\beta\mathbf{k}\times\mathbf{B}_0$ is perpendicular to $\mathbf{k}$ and since $\mathbf{k}$ and $\hbar\mathbf{v}_{\mathbf{k}}=\boldsymbol{\nabla}_{\mathbf{k}}\mathcal{E}$ are parallel, $\dot{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\mathcal{E}=0$, and the second term of (9) does not contribute. The left-hand side of (7) is now

$$\frac{\partial f_0}{\partial\mathcal{E}}\left\{\frac{\partial g}{\partial t}+[\mathbf{v}\cdot\mathbf{q}-i\dot{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}+\gamma_0B_{\text{eff}}]g\right\}. \quad (10)$$

The factor $\mathfrak{N}_{\mathbf{k}}$ may be written

$$\mathfrak{N}_{\mathbf{k}}=n_{\mathbf{k}\uparrow}-n_{\mathbf{k}+\mathbf{q}\downarrow}=n_{\mathbf{k}\uparrow}-n_{\mathbf{k}\downarrow}+n_{\mathbf{k}\downarrow}-n_{\mathbf{k}+\mathbf{q}\downarrow}. \quad (11)$$

We use the equilibrium distribution $f_0(\mathcal{E}_{\mathbf{k}})$ for $n_{\mathbf{k}}$ and obtain

$$\mathfrak{N}_{\mathbf{k}}=\frac{\partial f_0}{\partial\mathcal{E}}(\mathcal{E}_{\mathbf{k}\uparrow}-\mathcal{E}_{\mathbf{k}+\mathbf{q}\downarrow}). \quad (12)$$

We can divide (7) by the common factor $\partial f_0/\partial\mathcal{E}$ to obtain

$$\frac{\partial g_{\mathbf{k}}}{\partial t}+[\mathbf{v}\cdot\mathbf{q}+\gamma_0B_{\text{eff}}]\left(g_{\mathbf{k}}-\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\frac{\partial f_0}{\partial\mathcal{E}'}g_{\mathbf{k}'}\right)-\dot{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}g$$
$$=\tfrac{1}{2}\gamma_0[\mathbf{v}\cdot\mathbf{q}+\gamma_0B_{\text{eff}}]h, \quad (13)$$

and since $\partial f_0/\partial E\cong-\delta(E-E_F)$ the sum over $g_{\mathbf{k}'}$ is just $\delta\epsilon_2$ in the notation of Ref. 4. We may use (11) and (12) in (8) to obtain B_{eff}:

$$\gamma_0B_{\text{eff}}=\gamma_0B+\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\frac{\partial f_0}{\partial\mathcal{E}'}\gamma_0B_{\text{eff}}$$
$$=\frac{\gamma_0B}{1+\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\delta(\mathcal{E}'-\mathcal{E}_F)}\equiv\frac{\gamma_0B}{1+B_0}, \quad (14)$$

again in the notation of Ref. 4. In Sec. III we reexamine this equation in greater detail.

Our final term $-\dot{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}g$ is the term in which the Boltzmann-like transport equation and the self-consistent field equations differ.

This orbital term is

$$i(e/c)\mathbf{v}\times\mathbf{B}\cdot\boldsymbol{\nabla}_{\mathbf{k}}(g+\delta\epsilon_2), \quad (15)$$

in Ref. 4, whereas we have obtained

$$ih(e/mc)\mathbf{k}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}g. \tag{16}$$

Now,

$$\mathbf{v}=\frac{\hbar\mathbf{k}}{m}-h^{-1}\nabla_{\mathbf{k}}\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')n_{\mathbf{k}'}.$$

We can therefore write (16) as

$$i\frac{e}{c}\mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}g+\frac{e}{ch}[\nabla_{\mathbf{k}}\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')n_{\mathbf{k}'}]\times\mathbf{B}\cdot\nabla_{\mathbf{k}}g. \tag{17}$$

The difference (15)$-$(16) is thus

$$i\frac{e}{c}\Big\{\mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\frac{\partial f_0}{\partial\mathcal{E}'}g_{\mathbf{k}'}-h^{-1}\nabla_{\mathbf{k}}\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')n_{\mathbf{k}'}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}g\Big\}. \tag{18}$$

It is convenient to restore the factor $\partial f_0/\partial\mathcal{E}$, and if we assume $V(\mathbf{k},\mathbf{k}')=V(\mathbf{k}-\mathbf{k}')$,

$$\frac{ie}{\hbar c}\{\nabla_{\mathbf{k}}n\times\mathbf{B}\cdot\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\nabla_{\mathbf{k}'}\mathfrak{N}_{\mathbf{k}'}-\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\nabla_{\mathbf{k}'}n_{\mathbf{k}'}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}\mathfrak{N}_{\mathbf{k}}\}. \tag{19}$$

These two terms differ only in the roles of $\mathbf{k}$ and $\mathbf{k}'$. The angular average value of the difference, that is (19) summed over $\mathbf{k}$, clearly vanishes, but higher moments do not, and, to the extent that these higher moments are involved in the solutions of (7), the two approaches will differ.

III. OTHER POTENTIALS

Because the interaction function $V(\mathbf{k},\mathbf{k}')$ appears prominently in the transport equations, the solutions of the equations fitted to experimental results yield information about the interactions. The experimental data are very meager, however. One or at best two, numbers are determined.[1] In Ref. 4 these experimental numbers are related to theoretical parameters describing a particular, assumed form for the interaction function:

$$V(\mathbf{k},\mathbf{k}')=V_0+V_1\cos\theta, \tag{20}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. This function is well behaved and smooth throughout the entire range of $\mathbf{k}-\mathbf{k}'$. We shall consider here the extent to which this interpretation is demanded by the data and the possible consequences of including less well-behaved functions in the parametrizations.

The interactions appear in Eq. (3) in two ways, explicitly in the driving field and response terms, and implicitly in the energy and occupation differences of spin up and down states.

We shall treat two different forms of interaction potential. The first is similar to (20),

$$V=\tilde{V}_0+\tilde{V}_1(\mathbf{k}-\mathbf{k}')^2. \tag{21}$$

The second is the Coulomb interaction

$$V=U/(\mathbf{k}-\mathbf{k}')^2. \tag{22}$$

In the preceding section, we incorporated the exchange energy and the kinetic energy into a single function, assumed differentiable, in order to make at least partial contact with the Silin equation.

We wish to consider cases where the assumption of differentiability fails. We therefore study Eq. (3), in which the assumption is unnecessary, rather than (7) where it is required.

Our first task is to evaluate $\mathfrak{N}=n_{\mathbf{k}+\mathbf{q}\downarrow}-n_{\mathbf{k}\uparrow}$. We make a 0 temperature approximation in which $n_{\mathbf{k},\sigma}=1$ for $\mathcal{E}_{\mathbf{k},\sigma}<\mathcal{E}_F$, $=0$ for $\mathcal{E}_{\mathbf{k},\sigma}>\mathcal{E}_F$.

The single-particle energy of the electrons is

$$\mathcal{E}_{\mathbf{k},\sigma}=\frac{h^2\mathbf{k}^2}{2m}-\sum_{\mathbf{k}'}{}'V(\mathbf{k},\mathbf{k}')n_{\mathbf{k}',\sigma}+\tfrac{1}{2}\gamma_0\mathbf{B}\cdot\boldsymbol{\sigma}. \tag{23}$$

Using (21), we have

$$\mathcal{E}_{\mathbf{k},\sigma}=\frac{h^2\mathbf{k}^2}{2m}-\frac{k_{F,\sigma}{}^3}{6\pi^2}[\tilde{V}_0+\tilde{V}_1(\mathbf{k}^2+\tfrac{3}{5}k_{F,\sigma}{}^2)]+\tfrac{1}{2}\gamma_0\mathbf{B}\cdot\boldsymbol{\sigma}. \tag{24}$$

Since the Fermi energy must be the same for both spin species, we can use (24) to obtain the difference of the two Fermi radii $k_{F\downarrow}-k_{F\uparrow}=\Delta k_F$:

$$\Delta k_F{}^{(1)}=\frac{\gamma_0 B}{(h^2k_F/m)}\left[1-\frac{k_Fm}{2\pi^2h^2}\left(\tilde{V}_0+\frac{8}{3}\tilde{V}_1k_F{}^2\right)\right]^{-1}. \tag{25}$$

The corresponding result for the potential in Eq. (22) is

$$\Delta k_F{}^{(2)}=\frac{\gamma_0 B}{(h^2k_F/m)}\left[1-\frac{mU}{4\pi^2h^2k_F}\right]^{-1}. \tag{26}$$

To obtain $\mathfrak{N}$, then, we need to compare two Fermi spheres of slightly different radii, the larger one displaced by $-\mathbf{q}$. However, rather than treat the geometry of this situation exactly, we shall approximate $\mathfrak{N}$ with a product of a δ function on the spherical surface and a weighting factor over the surface which gives the same value for

$$\int_{|\mathbf{k}|<k_F}^{|\mathbf{k}|>k_F}\mathfrak{N}_{\mathbf{k}}d|\mathbf{k}|.$$

Thus we use

$$\mathfrak{N}_{\mathbf{k}}=\delta(|\mathbf{k}|-k_F)\left(\Delta k_F-\frac{\mathbf{k}\cdot\mathbf{q}}{k_F}\right). \tag{27}$$

It is clear from the form of Eq. (3) that $\mathfrak{N}_{\mathbf{k}}$ has essentially the same behavior as a function of $|\mathbf{k}|$ as $\mathfrak{N}_{\mathbf{k}}$, so

that we are led to take

$$\mathfrak{M}_k=\delta(|\mathbf{k}|-k_F)\times\{m_0+m_{1x}\mathbf{k}_x+m_{1y}\mathbf{k}_y+m_{1z}\mathbf{k}_z+\cdots\}. \quad (28)$$

In the course of using this expansion in (3) to obtain the coefficients of $\mathfrak{M}$, one observes that the each successively higher-order coefficient is also of higher order in qR_c, where R_c is the cyclotron radius. (See Sec. IV, however.) This circumstance greatly facilitates solution; (28) proves to be adequate as it stands to obtain the spin-wave spectrum accurate to terms in $q^2R_c^2$. All terms of relative order q/k_F in (3) have also been dropped since $q/k_F\ll qR_c\ll 1$ in the circumstances of the experiments.[1]

We evaluate the integral in (3) in terms of m_0 and m_{1i} to obtain

$$-\frac{\delta(|\mathbf{k}|-k_F)}{2\pi^2}[\tilde{V}_0+(8/3)k_F^2\tilde{V}_1]k_F^2\times\left(m_0\frac{\mathbf{k}\cdot\mathbf{q}}{k_F}+\Delta k_F[m_{1x}k_x+m_{1y}k_y+m_{1z}k_z]\right) \quad (29)$$

for the case of potential (21). Using the potential (22) gives the same expression but with

$$[\tilde{V}_0+(8/3)k_F^2\tilde{V}_1] \quad \text{replaced by} \quad \frac{U}{2k_F^2}.$$

We have seen that Δk_F is given in terms of $\tilde{V}_0+(8/3)\tilde{V}_1k_F^2$. We notice that this same combination has occurred in (29), and this exhausts the dependence of the terms in (3) on the interactions. It is therefore clear that the use of the isotropic potential (21) provides only a single parameter for fitting the experiments and that this is the same parameter as appears in the static paramagnetism. With the parametrization of (21), we can, therefore, interpret the spin-wave spectroscopy of Ref. 1 as a measurement of the exchange enhancement of the spin susceptibility of heretofore unattainable precision.

However, the assumption that $V(\mathbf{k},\mathbf{k}')$ depends only on $|\mathbf{k}-\mathbf{k}'|$ is a restrictive one. We have no reason to suppose that the variation of V should be the same for $\mathbf{k},\mathbf{k}'$ separations both parallel and perpendicular to the Fermi surface. In fact, we need only cite the Frölich electron-phonon-electron coupling to give a counter example. We shall not treat such potentials here, since our whole development has incorporated the assumption that V is a function only of $\mathbf{k}-\mathbf{k}'$.

Still it is clear that Δk_F is determined by the "perpendicular" aspect of the interactions and the exchange integral of (3) by the "parallel" aspect.

Substituting into Eq. (3) and solving for ω yields

$$\omega=\omega_s+\frac{1}{3}\left(\frac{\hbar k_F}{m}\right)^2q^2(1-\beta)\frac{\omega_c^2\cos^2\Delta-\beta^2\Omega_0^2}{(\Omega_0^2\beta^2-\omega_c^2)\beta\Omega_0}, \quad (30)$$

in which

$$\hbar\omega_s\equiv\gamma_0B, \quad (31)$$

$$\beta\equiv\frac{mk_F}{2\pi^2\hbar^2}[\tilde{V}_0+(8/3)k_F^2\tilde{V}_1]=\frac{mk_F}{2\pi^2\hbar^2}(V_0-\tfrac{1}{3}V_1), \quad (32)$$

$$\omega_c\equiv eB/mc, \quad (33)$$

$$\Omega_0\equiv\gamma_0B/(1-\beta)=\omega_s/(1-\beta), \quad (34)$$

Δ is the angle between $\mathbf{q}$ and $\mathbf{B}$, and m is the cyclotron mass. In (32) we have shown the results in terms of the potential (20). Potentials (20) and (21) are identical on the Fermi surface if

$$V_0=\tilde{V}_0+2k_F^2\tilde{V}_1 \quad \text{and} \quad V_1=-2k_F^2\tilde{V}_1. \quad (35)$$

Two experimental quantities are measured[1]: the angle Δ_c between field and the normal to the sample for which the dispersion of the spin waves vanishes and all peaks coalesce into the main CESR line, and the frequency splitting of the first resonance at some reference angle, in this case $\Delta=\frac{1}{2}\pi$. The angle of vanishing dispersion[1]$=69.7°$. Inserting this into (30) and using $\omega_s/\omega_c=1.24$ one obtains $\beta=0.219$. Using this value and $k_F=0.92\times10^8$ cm^{-1} and $q=133$, one obtains for $H_0-H_1=17.5$ G. The experimental value is about 18.2 G more or less; the line is somewhat broad. This 3 to 4% discrepancy could be caused by a number of things: an error in Δ_c of 0.8°, which is twice the quoted experimental range, an error of $0.01\rightarrow0.02\times10^{-8}$ in k_F or the same fractional error in q (which is unlikely) or m. The theory could require modification by inclusion of the effect of higher terms in the expansion for $\mathfrak{M}_\mathbf{k}$ or $V(\mathbf{k},\mathbf{k}')$ since the value of the expansion parameter[1] is $qR_c\approx0.1$, or by inclusion of the possible effects of anisotropic potentials $V(\mathbf{k},\mathbf{k}')$ which depend on $\mathbf{k}+\mathbf{k}'$. The discrepancy is not clearly significant; we have achieved a fairly good one-parameter interpretation of the experiments.

The use of the Coulomb potential, Eq. (22), with $U=4\pi e^2$ does not give satisfactory agreement:

$$\frac{m(4\pi e^2)}{4\pi^2\hbar^2k_F}=0.8113,$$

which is too large by a factor of almost 4. The inclusion of phonon effects in addition to the naked Coulomb repulsion would be expected to lower this somewhat, but not nearly by the required 70%. Still it is interesting that unscreened Coulomb interactions produce no greater discrepancy.

In summary, we have seen that the experimental results demand that (1) the Coulomb interactions between the electrons are somehow screened, (2) the value of the interaction parameter β must be $0.22\pm$ a few percent, and (3) any parametrization of the interaction function for which the singular behavior, if any,

is no worse than Coulombic or any linear combination of such potentials which produce a net $\beta=0.22$ will serve to interpret the experiments.

There are theoretical reasons to expect the potentials to be nonsingular, or at least not so strongly singular as the Coulomb interaction. This conclusion is neither sustained nor refuted by these experiments. We do expect the inclusion of the electron-phonon-electron interaction to require treating singular potentials. We believe that the experimental determination of the form of the interactions is an independent problem of equal importance with the theoretical study and are disappointed that these experiments provide no test of the question.

IV. UNCERTAINTY PRINCIPLE

The solution of the transport equations—ours and Silin's—involves an expansion in powers of

$$qR_c \ll 1, \tag{36}$$

where R_c is the radius of the cyclotron orbit, and this range of wave vectors and magnetic fields is explored in the experiments. The derivations of both transport equations require the clear distinction between occupation of states $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$, at least along the $\mathbf{k}$-space direction defined by $\mathbf{q}$. We have made this condition explicit in Ref. 8; Silin's use of the Boltzmann equation seems to imply it; to write $\dot{\mathbf{v}}\cdot\nabla_k f$ assumes that $\dot{\mathbf{v}}$ and $\mathbf{k}$ may be sufficiently precisely stated to make the expression meaningful. We must be able to distinguish the velocity and its time derivative for the two states. Along a direction perpendicular to $\mathbf{q}$, we need only require that the blurring of precise occupations not be so severe as to destroy the Fermi surface. Hence this second assumption is contained implicitly in the use of δ functions on the Fermi energy for the characteristic functions $n_{\mathbf{k}+\mathbf{q}\downarrow}-n_{\mathbf{k}\uparrow}$ as they appear [Eqs. (2), (9), (12), (13), and so forth]. Thus, we require

$$(\Delta k_{\parallel})(\Delta k_{\perp}) \ll qk_F \tag{37}$$

as the condition for the applicability of the semiclassical Boltzmann equation theory of electron plasma response. We must emphasize, however, that the condition (37) limits only the conceptual tools with which the transport equation has been constructed and that we do not know that the final equation is invalid in the range $q \lesssim (\Delta k_{\parallel})$, only that the derivation fails. Still, to attach any significance to the interpretations of experiments will require a new derivation whose applicability is not open to question.

We have derived a condition (Ref. 8) similar to (37); the meanings of the Δk's are changed somewhat.

The condition (37) or the similar condition in Ref. 8 contradicts (36). Let us see how this comes about.

The electron states in the Boltzmann-equation approach are described by wave packets acted on by the Lorentz force—in particular $e\mathbf{v}\times\mathbf{B}$—with $\mathbf{v}$ given by $\hbar^{-1}\nabla_{\mathbf{k}}\mathcal{E}(\mathbf{k})$. The packets are reduced in spread in $\mathbf{k}$ space to a well-defined $\mathbf{k}$ for the construction of a many-body wave function. In the absence of a magnetic field, $\mathbf{v}_{\mathbf{k}}$ thereby becomes progressively better defined.

In the presence of a magnetic field, however, $\mathbf{v}_{\mathbf{k}}$ is less well defined by this process. The reason is that $m\mathbf{v}=\mathbf{p}-e\mathbf{A}/c$ depends on position $\mathbf{r}$ as well as momentum, and as the states are progressively better defined in momentum space, they spread out in direct space, making operators like $\mathbf{r}$, $\mathbf{v}$, and ∇ uncertain. The expression

$$\mathbf{v}\times\mathbf{B}\cdot\nabla_{\mathbf{k}}g_{\mathbf{k}} \tag{38}$$

has meaning only if we can tolerate an uncertainty in $\mathbf{v}$.

The two components of $\mathbf{v}$ of interest are

$$mv_x = p_x + (eB/2c)y, \tag{39}$$

$$mv_y = p_y - (eB/2c)x. \tag{40}$$

If we calculate the commutator of these operators, we obtain

$$[v_x, mv_y] = i\hbar(eB/mc) = i\hbar\omega_c. \tag{41}$$

Therefore, we must have

$$(\Delta v_x)(\Delta v_y) \gtrsim \hbar\omega_c/m, \tag{42}$$

or in terms of k's

$$\hbar^2(\Delta k_x)(\Delta k_y)/2m \gtrsim \tfrac{1}{2}\hbar\omega_c. \tag{43}$$

If we take $\mathbf{q}$ to lie along the x direction, we have Δk_x for $\Delta k_{\parallel}$ and Δk_y for $\Delta k_{\perp}$. Thus,

$$\hbar^2 qk_F/2m > \hbar^2(\Delta k_{\parallel})(\Delta k_{\perp})/2m \gtrsim \tfrac{1}{2}\hbar\omega_c \tag{44}$$

defines the limiting range of the Boltzmann equation in magnetic fields. Now,

$$R_c = \hbar k_F/m\omega_c. \tag{45}$$

We have then for (44)

$$\hbar^2 qk_F/2m > \tfrac{1}{2}\hbar\omega_c = \hbar^2 k_F/2mR_c, \tag{46}$$

from which we obtain

$$qR_c > 1. \tag{47}$$

The same condition is obtained from more formal arguments in Ref. 8.

V. CONCLUSIONS

We have argued previously for a certain transport equation. We have shown here that this equation is different from the one suggested by Silin and used by Platzman and Wolff. We have calculated the CESR spin-wave satellite spectrum using our transport equation.

The electron interaction function, $V(\mathbf{k},\mathbf{k}')$ cannot be uniquely determined from the present experiments since different functional forms for V may lead to the

same experimental results. Even strongly singular functions $V(\mathbf{k},\mathbf{k}')$ may be quite consistent with the experimental results.

If our transport equation is applicable, the experiments represent a measurement of Pauli paramagnetism of enormously improved precision. However, the derivations of the transport equations are uncertain in the range of experiments so far carried out, raising doubts about the significance of the numbers which either theory inexorably produces.

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Impurity Ionization in Germanium in Strong Magnetic Fields*

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Low-temperature electrical breakdown in p-type germanium has been investigated at magnetic-field strengths up to 52 kOe. The electric-field strength required for the onset of breakdown in gallium-doped germanium increased from 4.2 V/cm at zero magnetic field to 22 V/cm in a 50-kOe transverse magnetic field. For a longitudinal magnetic field of the same range, there is a smaller variation of breakdown electric field with a maximum of 7.2 V/cm at 50 kOe. The effect is independent of the polarity of electric field applied to the specimen in all cases. The increase in electric field required for breakdown appears to be in reasonable agreement with recent theories of impact-ionization phenomena, provided that the quantization and shift of energy levels in a strong magnetic field are considered together with the effect of transverse magnetic fields on the rate at which carriers gain energy from an electric field. Measurements of the changes of the far-infrared photoconductivity of the specimen in the magnetic field provide further evidence of the effect of the field on the ionization phenomena.

INTRODUCTION

THE low-temperature electrical-breakdown effect in germanium containing group-III impurities has been investigated, both experimentally and theoretically.[1–4] There is fair qualitative agreement between the experimental results and the theoretical descriptions of the effect in most aspects. The nature of the breakdown phenomena has been experimentally investigated as a function of a variety of experimental parameters by both Sclar and Burstein[1] and by Zavaritskaya.[2] However, the effect of a magnetic field on the impact-ionization process has been investigated only in the low-field regime[1,2,5] where quantization effects are not important. This paper reports measurements of low-temperature electrical breakdown in strong magnetic fields where quantization and shift of the energy levels are significant. The results are interpreted in terms of recent theories of breakdown phenomena taking into account the effect of the magnetic field on the energy-level separation, the mobility and the recombination coefficient. Measurements of the effect of the magnetic field on the far-infrared extrinsic photoconductivity of the same specimens are used to derive supporting evidence as to the nature of the physical phenomena involved.

Low-temperature electric breakdown in impurity semiconductors is caused by impact ionization of neutral impurity centers by hot carriers.[6] As the strength of the applied electric field increases, the average kinetic energy of the charge carriers becomes sufficient to ionize neutral centers. The recombination process which reduces the number of free carriers becomes less effective at high fields since the capture cross section decreases with increasing carrier energy. For electric-field intensities exceeding a certain critical value, the ionization rate exceeds the recombination rate and a nonequilibrium condition is obtained. The critical field is regarded as the breakdown field with the breakdown condition being expressed in terms of the kinetic equation[7]

$$dp/dt = A_T(N_A - N_D - p) + p[A_I(N_A - N_D - p) - B_T(N_D + p)] - p^2 B_I(N_D + p). \quad (1)$$

The increase in the number of free holes p is related to A_T and A_I, which represent the rates of carrier generation by thermal processes and impact ionization, respectively; B_T and B_I describe the thermal and Auger recombination rates, respectively. In a compensated material containing both types of impurities, N_D is the density of donor centers, while N_A is the acceptor density. Under steady-state conditions, if we neglect Auger recombination and thermal-carrier generation

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